Ex

$$x'' + 16x = \cos 4t$$

$$s^2Y(s) - sy_0 - y_0' + 16\,Y(s) = \frac{1\;s}{s^2+16}$$

$$(s^2+16)Y(s) = 1 + \frac{s}{s^2+16}$$

$$Y(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}$$

$$y = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t.$$

EX.

$$y'''(t) + 3y'(t) = 9t^2 - 12t + 6 \;;\; y(0)=3;\; y'(0)=0;\; y''(0)=-4$$

$$s^3Y(s) - 3s^2 + 4 + 3sY(s) - 9 = \frac{18}{s^3} - \frac{12}{s^2} + \frac{6}{s}$$

$$Y(s) = \frac{\cancel{(s^2+3)}(3s^2-4s+6)}{s^4\cancel{(s^2+3)}}$$

$$Y(s) = \frac{3}{s} - \frac{4}{s^2} + \frac{6}{s^4}$$

$$y = 3 - 2t^2 + t^3$$

EX. the motion of a mass on a horizontal plane can be described by the IVP

$$x'' + 48x = 0 \qquad x_0 = 1/4 \;;\; x_0' = 0$$

$$s^2X(s) - s\frac{1}{4} + 48^2X(s) = 0$$

$$X(s) = \frac{1}{4}\,\frac{+s}{s^2+48^2}$$

$$x(t) = \frac{1}{4}\cos 48t.$$

DEFINITION OF A BEAM.

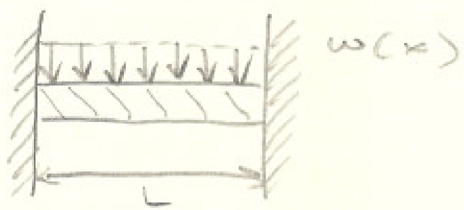
the static definition of a beam (uniform) of length L, carrying a uniform load of $w(x)$ per unit legth is found from the 4th order DE.

$$EI\frac{d^4y}{dx^4} = w(x)$$

E: Young's modulus

I: moment of inertia

EX. A beam of legth L is well embedded a both ends as shown below



In this case, the definition $y(x)$ must satisfy the following definitions

$y_0 = (0)$ $\quad y_0' = 0$

$y(L) = 0$ $\quad y'(L) = 0$

The first two conditions indicate that there is no vertical definition at the ends, the last 2 conditions mean that along the x axis, the slope is zero at both ends. find the definition of the beam when a constant load w_0 is uniformly distributed across its legth.

ie $\quad w_0(x) = C \quad , 0 < x < L$

so we have

$$EI\frac{d^4y}{dx^4} = C$$

LT both sides

$$EI\left[s^4 Y(s) - s^3 y_0 - s^2 y_0' - s y_0'' - y_0'''\right] = \frac{C}{s}$$

rearrange

$$s^4 Y(s) - s y_0'' - y_0''' = \frac{C}{EIs}$$

let $c_1 = y_0''$; $c_2 = y_0'''$

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{C}{EI s^5}$$

$$y(x) = \frac{c_1}{2!}\mathcal{L}^{-1}\left[\frac{2!}{s^3}\right] + \frac{c_2}{3!}\mathcal{L}^{-1}\left[\frac{3!}{s^4}\right] + \frac{C}{IE4!}\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right]$$

$$= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3 + \frac{C}{24EI}x^4$$

DERIVATIVES OF TRANSFORMS.

for $n = 1, 2, 3, \ldots$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\left[\mathcal{L} f(t)\right]$$

EX.

$$\mathcal{L}\left[t e^{3t}\right] = (-1)^1 \frac{d^1}{ds^1}\left[\mathcal{L} e^{3t}\right]$$

$$= \frac{1}{(s-3)^2}$$

Called their convolation and is decreased by $f*g$. The convolation has the property that the Laplace transform of $f*g$ has Laplace transform of f multiplied by the Laplace transform g

defined on $[0,\infty]$, then the convolation $f*g$ is the function defined by

$$f*g = \int_0^t f(t-\tau)\, g(t)\, dt$$

<u>Proof</u>

let $f(s) = \mathcal{L} f(t)$

$G(s) = \mathcal{L}\, g(t)$

as usual from the definition of the transform we have.

$$f(s)\, G(s)$$

Replace t to τ as the variable of inigration in the last integral, then

$$F(s)\, G(s) = \int_0^\infty F(s) e^{-s\tau}\, g(\tau)$$

we have

$$F(s)\, G(s) = \int_0^\infty \mathcal{L}[f(t-\tau)\, u(t-\tau)]\, g(\tau) d\tau$$

but we have

$$= \mathcal{L}[f(t-\tau)\, u(t-\tau)]$$

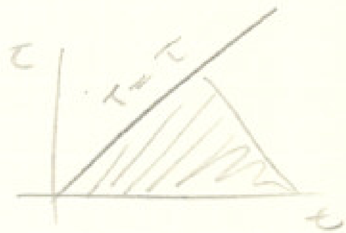
$$= \int_0^\infty e^{-st} f(t-\tau)\, u(t-\tau)\, dt$$

$$F(s)G(s)=\int_0^\infty\int_0^\infty e^{-st}g(\tau)u(t-\tau)f(t-\tau)\,dt\,d\tau$$

we know that

$$u(t-\tau)=\begin{cases}0 & t<\tau\\ 1 & t\geq\tau\end{cases}$$

$$\therefore\quad F(s)G(s)=\int_0^\infty\int_\tau^\infty e^{-st}g(\tau)f(t-\tau)\,dt\cdot d\tau$$



In the diagram, it shows the $t\tau$ plane the last integration is over, the shaded are which consists of points (t,τ) satisfying $0<\tau\leq t<\infty$

Reserve the order of integration in the last integral, we get

$$F(s)G(s)=\int_0^\infty\int_0^t e^{-st}g(\tau)f(t-\tau)\,d\tau\,dt$$

$$=\int_0^\infty e^{-st}\left[\int_0^t g(\tau)f(t-\tau)\,d\tau\right]dt$$

$$=\int_0^\infty e^{-st}(f*g)\,dt$$

$$=\mathcal{L}(f*g)$$

The convolution theroem has an inverse version

$$\mathcal{L}^{-1}[F(s)G(s)]=f*g$$

EX find the inverse LT of the product given below by the convolution theroem

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s+4)}\right]$$

note: ~~one~~ may of course find the inverse using partial fractions.... but....

$F(s) = \dfrac{1}{s-1}$ $\qquad$ $G(s) = \dfrac{1}{s+4}$

we know that

$f(t) = e^{t}$ $\qquad$ $g(t) = e^{-4t}$.

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s+4)}\right] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= \int_0^{t} e^{\tau} e^{-4(t-\tau)}\,d\tau$$

$$= e^{-4t}\int_0^t e^{\tau} e^{4\tau}\,d\tau$$

$$= e^{-4t}\left[\frac{1}{5}e^{5\tau}\right]_0^t$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}$$

which is the same as partial fractions.

EX. Solve the IVP by L.T.

$$y''(t) + 100y(t) = 100 \quad \text{for } y(0) = 2 \quad y'(0)$$

L.T. both sides.

$$s^2Y - 2s + 100Y = \frac{100}{s}$$

$$Y = \frac{2s}{s^2+100} + 10\left\{\frac{1}{s}\cdot\frac{10}{(s^2+10)}\right\}$$

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2+100}\right] = 2\cos 10t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{10}{s^2+10}\right] = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 = g(s) \qquad \mathcal{L}^{-1}\left[\frac{10}{s^2+10}\right] = \sin 10t = f(s)$$

$$= \int_0^t \sin 10\tau \, d\tau = -\cos 10\tau \Big|_0^t = 1 - \cos 10t$$

$$y(s) = 1 + \cos 10t$$

2×2 SYSTEMS. (L.T. METHOD)

Consider $\frac{dx}{dt} = y$; $\frac{dy}{dt} = 5y - 6x + 2e^{-t}$

solve for $x(t)$ & $y(t)$ for $t > 0$

LT Both sides.

$$Xs = Y \quad ; \quad sY = 5Y - 6X + \frac{2}{s+1}$$

substitude for Y and solve

$$X = \frac{s+3}{(s-3)(s+2)(s+1)}$$

Partial fractions yeilds

$$X(s) = \frac{3}{2}\frac{1}{(s-3)} - \frac{5}{3}\frac{1}{(s-2)} + \frac{1}{6}\frac{1}{(s+1)}$$

$$x(t) = \frac{3}{2}e^{3t} - \frac{5}{3}e^{2t} + \frac{1}{6}e^{-t}$$

now we need to solve for $y(t)$

$$Y(s) = \frac{s^2+3s}{(s-3)(s-2)(s+1)}$$

EX Solve the IVP given by.

$$x'(t) + x(t) + y'(t) - y(t) = 2$$
$$x''(t) + x'(t) - y'(t) = \cos t$$

for $x(0) = 0$; $x'(0) = 2$; $y(0) = 1$

$$sX + X + sY - 1 - Y = \frac{2}{s}$$

$$s^2X - 2 + sX - sY - 1 = \frac{s}{s^2+1}$$

substitude an solve.

MATRIX ALEGERBRA

$$a_{11}x_1 + a_{21}x_2 = b_1$$
$$a_{12}x_1 + a_{22}x_2 = b_2$$

we write this in compact form.

vectors

$$\underline{A}\,\underline{X} = \underline{b}$$

matrix

$$\underline{A} = \underset{2\times 2}{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}} \qquad \underline{X} = \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underline{A}^{-1}\underline{A}\,\underline{X} = \underline{A}^{-1}\underline{b} = \underline{X}$$

MATRIX ALGERBRA.

introduction: a system such as.

$$\begin{array}{ccccccc}
a_{11}x_1 + & a_{12}x_2 & + & \cdots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 + & a_{22}x_2 & + & \cdots + & a_{2n}x_n & = & b_2 \\
\cdot\ + & \cdot & & \cdots + & \cdot & = & \cdot \\
\cdot\ + & \cdot & & \cdots + & \cdot & = & \cdot \\
\cdot\ + & \cdot & & \cdots + & \cdot & = & \cdot \\
a_{n1}x_1 + & a_{n2}x_2 & + & \cdots + & a_{nn}x_n & = & b_n
\end{array}$$

the above can be written in compact form

$$\underline{A}\,\underline{X} = \underline{b}$$

where,

$$\underline{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$n \times n$ matrix
rows colums.

$$\underline{b} = \begin{vmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{vmatrix} \quad \underline{X} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{vmatrix}.$$

$n \times 1$ matrix

both are vectors.

for $n = 2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

a_{11}, a_{22} are diagonal entrees

a_{12}, a_{21} are off diagonal entrees.

EX.

$$\underline{A} = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 6 \end{bmatrix}$$ is a rectangular matrix with dimensions of 2×3

note: two matrices are said to be equal if $a_{ij} = b_{ij}$ for all i & j.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

you can add matrices if their dimensions are the same dimensions.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & h+d \end{bmatrix}$$

SCALER MULTIPLE

of a matrix, If k is a real number, then the scaller multiple of a matrix $\underline{A}(n\times n)$

$$k\underline{A} = \begin{vmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \dots & \vdots \\ ka_{n1} & ka_{n2} & \dots & ka_{nn} \end{vmatrix}$$

EX.

$$5\begin{vmatrix} 2 & -3 \\ 4 & -1 \\ \frac{1}{5} & 6 \end{vmatrix} = \begin{vmatrix} 10 & -15 \\ 20 & -5 \\ 1 & 30 \end{vmatrix}$$

DIFFERENCE OF MATRICES.

$$\underline{A} + (-1)\underline{B} = A - B$$

PROPERTIES OF MATRIX ADDITION & SCALLER.

Suppose $\underline{A}$, $\underline{B}$, and $\underline{C}$ are $m\times n$ matrices and k_1 & k_2 are scallers.

$$\underline{A}+\underline{B}=\underline{B}+\underline{A}$$
$$\underline{A}+(\underline{B}+\underline{C})=(\underline{A}+\underline{B})+\underline{C}$$
$$k_1k_2\underline{A}=k_1(k_2\underline{A})$$
$$k_1(\underline{A}+\underline{B})=k_1\underline{A}+k_1\underline{B}$$
$$(k_1+k_2)\underline{A}=k_1\underline{A}+k_2\underline{A}$$

MATRIX MULTIPLYCATION

If $\underline{A}$ is an $m\times n$ matrix and $\underline{B}$ is a $n\times q$ matrix than the product

$\underline{AB}$ is a $m\times q$ matrix.

EX

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \quad ; \quad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$$

$$A \cdot B = \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \end{vmatrix}$$

EX,

$$A = \begin{vmatrix} 4 & 7 \\ 3 & 5 \end{vmatrix} \qquad B = \begin{vmatrix} 9 & -2 \\ 6 & 8 \end{vmatrix} \qquad AB = \begin{vmatrix} 78 & 48 \\ 57 & 34 \end{vmatrix}$$

$$BA = \begin{vmatrix} 30 & 53 \\ 48 & 82 \end{vmatrix}$$

we see that

$$AB \neq BA$$

ASSOCIATE LAW

$$\underbrace{\underset{m\times r}{A}(\underset{r\times p}{B}\,\underset{p\times n}{C})}_{m\times n} = \underbrace{(AB)C}_{\text{same}}$$

DISTRIBUTED LAW

If B & C are both $(r \times n)$ matrices, and A is an $(m \times r)$ matrix, then.

$$A(B + C) = AB + AC$$

note that if the product given by $(B + C)A$ is defined, then it is equal $BA + CA$

TRANSPOSE.

If A is a $(m \times n)$ matrix, then, A^T is a $n \times m$ matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{vmatrix} \qquad A^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{mn} \end{vmatrix}$$

EX.

$$\underline{B} = \begin{vmatrix} 5 \\ 0 \\ 3 \end{vmatrix} \qquad \underline{B}^T = \begin{vmatrix} 5 & 0 & 3 \end{vmatrix}$$

flip on this line... rows become columns, vica versa.

PROPERTIES OF TRANSPOSE

$$(\underline{A}^T)^T = \underline{A}$$
$$(\underline{A} + \underline{B})^T = \underline{A}^T + \underline{B}^T$$
$$(A\underline{B})^T = \underline{B}^T \underline{A}^T$$
$$(k\underline{A})^T = k\underline{A}^T$$

SPECIAL MATRICES.

Zero or **Null** matrix ($\underline{0}$)

EX

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = 0 \\ a_{21}x_1 + a_{22}x_2 = 0 \end{aligned}\right\rangle \quad \underline{A}\,\underline{x} = \underline{0}$$

a null matrix can have various kinds of dimensions.

note:

$$\underbrace{\underline{A} + \underline{0}}_{\text{same dimension}} = \underline{A} \qquad \underbrace{\underline{A} + (-1)\underline{A}}_{A \text{ is square}} = 0$$

TRIANGULAR MATRIX.

EX.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

upper triangular matrix.

DIAGNOL MATRIX

all values are zero, except diagrol element.

EX.
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

UNIT MATRIX

an $n \times m$ matrix which is a diagonal matrix where all its elements are zero except the diagonal which is 1.

$$\underline{I} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

interestingly

$$\underline{I}\,\underset{3\times 3}{\underline{A}} = \underline{A}\,\underline{I}$$

SYMETRIC MATRIX

$$A = A^T$$

SYSTEMS OF LINEAR EQUATIONS.

A system of n linear algerbreaic equations. are given by, for $n = 3$

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{21} x_1 + a_{32} x_2 + a_{32} x_3 = b_3$$

can be written as $\underline{A}\underline{x} = \underline{b}$; if $\underline{b} = \underline{0}$, then it is known as homogeneous. algerbraic eqo.

if $\underline{b} \neq \underline{0}$ then it is non-homogeneous.

EX solve

$$3x_1 + 6x_2 = 3$$
$$x_1 - 4x_2 = 7$$

$$\underline{A} = \begin{vmatrix} 3 & 6 \\ 1 & -4 \end{vmatrix} \quad \underline{b} = \begin{vmatrix} 3 \\ 7 \end{vmatrix} \quad \underline{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

A system of linear equations can be transformed into an equivilent system by the following steps.

multiply by an equal non zero constant.

Introduce the position of equations. in the system.

add a non zero multiple of one equation to any other equation

(then mersour ckunched numbers for awhile).

AUGUMENTED MATRIX.

a set of n linear algerbraic eqs

$\underline{A}\,\underline{x} = \underline{b}$ in compact form

The above can be represented by an augumented matrix

$[\underline{A} \mid \underline{b}]$

GAUSSIAN ELIMINATION

Solve the set of linear equations by gaussian elimination.

EX.
$$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x^3 &= 9 \end{aligned}$$

SOLUTION:

Write the above set in augmented matrix form. Then perform, successive row opperations

$$\underline{A}|\underline{b} = \left[\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]$$ exchange rows one and two including b_1 & b_2

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \\ \text{"ROW"} \\ -5R_1 + R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right] \quad -\tfrac{1}{2}R_2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & -3 & 1 & 14 \end{array}\right] \quad 3R_2 + R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1+(9/2) & 3(\tfrac{9}{2})+14 \end{array}\right] \quad 2/11\, R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad x_3 = 5$$

The above is
$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ x_2 + \tfrac{3}{2}x_3 &= 9/2 \\ x_3 &= 5 \end{aligned}$$

Substitute back to solve for x_1, x_2

$$\underline{x} = \begin{bmatrix} 10 \\ -3 \\ 5 \end{bmatrix}$$

GAUSS-JORDAN METHOD'S

The gaussian elimination process is continued. ie- further row opperations. We will eventually end up with a unit matrix augumented by the solution of the system of equations

Continuing from the previous Gausian Elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right] \overset{-2R_2+R_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} 4R_3+R_1 \\ \\ -3/2 R_2+R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$ identical solution.

DETERMINATES.

an $\underline{A}$ square matrix has a number known as a determinate (which is scaler)

$$\det \underline{A} = |\underline{A}|$$

EX

$$|\underline{A}| = \left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = a_{11}a_{22} - a_{12}a_{21}$$

EX

$$|\underline{A}| = \left| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right|$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The process can be expanded to a higher order matrices. The above process was to evaluate $|\underline{A}|$ along the first row

The cofactor for a_{11} is $\underline{C}_{11}$

$$\underline{C}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

The cofactor for a_{12} is $\underline{C}_{12}$

$$\underline{C}_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

The cofactor for a_{13} is $\underline{C}_{13}$

$$\underline{C}_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\therefore \; |\underline{A}| = a_{11}\underline{C}_{11} - a_{12}\cdot\underline{C}_{12} + a_{13}\underline{C}_{13}.$$

In general the cofactor of a_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The above process can be written as a checker board pattern as follows.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

THEOREM

If A is an $(n \times n)$ square matrix then

$$|\underline{A}^T| = |\underline{A}| \qquad \text{Ex.} \quad |\underline{A}^T| = \left|\begin{bmatrix} 5 & 3 \\ 7 & -4 \end{bmatrix}\right| = |A| = \left|\begin{bmatrix} 5 & 3 \\ 7 & -4 \end{bmatrix}\right|$$

$$-41 \quad = \quad -41$$

THEOREM.

If $\underline{A}$ is an $(n \times n)$ matrix and if two of it's rows or colomns are the same, then the determinete is zero.

EX

$$\underline{A} = \begin{bmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{bmatrix} \qquad \det \underline{A} = |\underline{A}| = 0$$

THEOREM.

If $\underline{B}$ is a matrix obtained by interchanging any 2 rows or columns of an $(n \times n)$ matrix then

$$|\underline{B}| = -|\underline{A}|$$

EX.

$$\left| \begin{bmatrix} 2 & 1 & 3 \\ 6 & 0 & 7 \\ 4 & -1 & 9 \end{bmatrix} \right| = - \left| \begin{bmatrix} 4 & -1 & 9 \\ 6 & 0 & 7 \\ 2 & 1 & 3 \end{bmatrix} \right|$$

THEOREM

If $\underline{A}$ and $\underline{B}$ are both $(n \times n)$ matrices then,

$$|\underline{A}\,\underline{B}| = |\underline{A}|\,|\underline{B}|$$

EX.

$$A = \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 3 & -4 \\ -3 & 5 \end{bmatrix}$$

$$|\underline{A}\,\underline{B}| = \left| \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -3 & 5 \end{bmatrix} \right| = \left| \begin{bmatrix} -12 & 22 \\ 6 & -9 \end{bmatrix} \right| = -24$$

$$|\underline{A}| = -8 \qquad |\underline{B}| = 3 \qquad \therefore \quad |\underline{A}|\,|\underline{B}| = -24$$

THEOREM

If $\underline{A}$ is an $(n \times n)$ upper or lower triangular matrix, then $\det \underline{A} = |\underline{A}| =$ the product of the diagonala elements

MATRIX INVERSION.

If $\underline{A}$ is an $(n \times n)$ matrix and $(\underline{B})$ is an $(n \times n)$ matrix such that

$$\underline{A}\,\underline{B} = \underline{B}\,\underline{A} = \underline{I}$$

then $\underline{B}$ must be the inverse of $\underline{A}$, and $\underline{B}$ is written as

$$\underline{B} = \underline{A}^{-1} \neq \frac{1}{\underline{A}}$$

EX.

$$\underline{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\underline{A}\,\underline{B} = \begin{bmatrix} (2)(1) + (1)(-1) & (2)(-1) + (1)(2) \\ (1)(1) + (1)(-1) & (1)(-1) + (1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In general if $\underline{A}$ & $\underline{A}^{-1}$ are $(n \times n)$ then $\underline{I}$ is $(n \times n)$

PROPERTIES OF INVERSE MATRICES.

(i) $\left[\underline{A}^{-1}\right]^{-1} = \underline{A}$

$(\underline{A}\,\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$

$\left[\underline{A}^{T}\right]^{-1} = \left[\underline{A}^{-1}\right]^{T}$

MATRIX INVERSION BY THE ADJOINT METHOD.

The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is known as the minor of a_{ij}. ie - the determinate of the $((n-1) \times (n-1))$ matrix obtained by deleting the the i^{th} row and j^{th} column of $\underline{A}$?

basically I make a knew matrix out of the origenal values. $(-1)^{i+j} M_{ij}$, Transpose It, then divide by the determinant.